Exercise 13

Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

Solution

$$|z - z_0| = R$$

Square both sides.

$$\begin{aligned}
R^2 &= |z - z_0|^2 \\
&= (z - z_0)(\overline{z - z_0}) \\
&= (z - z_0)(\bar{z} - \overline{z_0}) \\
&= z\bar{z} - z\overline{z_0} - z_0\bar{z} + z_0\overline{z_0} \\
&= |z|^2 - (z\overline{z_0} + \bar{z}z_0) + |z_0|^2 \\
&= |z|^2 - 2\left(\frac{z\overline{z_0} + \overline{z\overline{z_0}}}{2}\right) + |z_0|^2 \\
&= |z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2
\end{aligned}$$

Therefore, $|z - z_0| = R$ can be written as

$$|z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$